Application of He's Homotopy Perturbation Method to Stiff Systems of Ordinary Differential Equations

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We propose He's homotopy perturbation method (HPM) to solve stiff systems of ordinary differential equations. This method is very simple to be implemented. HPM is employed to compute an approximation or analytical solution of the stiff systems of linear and nonlinear ordinary differential equations.

Key words: Homotopy Perturbation Method; Stiff Systems; Systems of Differential Equations.

1. Introduction

In this paper, we obtain the analytical or numerical solutions for systems of ordinary differential equations by He's homotopy perturbation method (HPM). In many different fields of science and engineering, it is very important to obtain exact or numerical solutions of systems of nonlinear ordinary differential equations. It is well known that nonlinear phenomena are very common in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and chemical physics. Searching for exact and numerical solutions, especially for traveling wave solutions of nonlinear equations in mathematical physics, plays an important role in the soliton theory [1, 2].

Hirota [3] proposed some approaches to solve nonlinear equations such as Bäcklund transformation and Hirota's bilinear method. There are also other methods to solve nonlinear equations, e.g. the sine-cosine method [4], the homogeneous balance method [5, 6], the Riccati expansion method [7] and variational iteration method [8 – 11]. The homotopy perturbation method was first proposed [12] and further developed and improved by He [13 – 16]. The method yields a very rapid convergence of the solution series in most cases. The main application of HPM shows miraculous exactness and convenience compared to other methods. The homotopy perturbation method is explained in the following section.

HPM can solve a large class of nonlinear problems efficiently, accurately and easily. Usually, one iteration leads to high accuracy of the solution. Although the goal of HPM was to find a technique to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems, HPM was proposed to search for limit cycles or bifurcation curves of nonlinear equations [17]. In [18] a heuristic example was given to illustrate the basic idea of HPM and advantages over the δ-method. The method was also applied to solve boundary value problems [19] and Laplace transform [20]. The motivation of this paper is to illustrate the merits of the method in solving some systems of ordinary differential equations. The homotopy perturbation method is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. The availability of computer symbolic packages such as Mathematica and Maple gives a mathematical tool to perform some complicated manipulations and to do some modifications on a method for a specific problem easily.

The paper is organized as follows: in the following section the homotopy perturbation method is explained. In Section 3 we propose the solution method and we solve three test problems. Numerical results are reported in Section 4. Finally, the paper is concluded in Section 5.

2. Basic Idea of the Homotopy Perturbation Method

The homotopy perturbation method is a combination of the classical perturbation technique and the homotopy technique. He [14] considered the nonlinear differential equation

$$A(u) = f(r), \qquad r \in \Omega, \tag{1}$$

where $f(r)$ is a known analytical function. The operator A can be divided into two parts, M and N. Therefore (1) can be rewritten as

$$M(u) + N(u) = f(r). \tag{2}$$

He [12, 14] constructed a homotopy $v(r,p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$\begin{aligned}\mathcal{H}(v,p) = (1-p)\big[M(v) - M(y_0)\big] \\ + p\big[A(v) - f(r)\big] = 0,\end{aligned} \tag{3}$$

or

$$\begin{aligned}\mathcal{H}(v,p) = M(v) - M(y_0) + pM(y_0) \\ + p\big[N(v) - f(r)\big] = 0,\end{aligned} \tag{4}$$

where $r \in \Omega$, y_0 is an initial approximation of (1), and $p \in [0,1]$ is an imbedding parameter. Hence it is obvious that

$$\begin{aligned}\mathcal{H}(v,0) = M(v) - M(y_0) = 0, \\ \mathcal{H}(v,1) = A(v) - f(r) = 0.\end{aligned}$$

Changing of p from 0 to 1 causes that $\mathcal{H}(v,p)$ changes from $M(v) - M(y_0)$ to $A(v) - f(r)$. In topology, it is called deformation, and $M(v) - M(y_0)$ and $A(v) - f(r)$ are called homotopics. By applying the perturbation technique we can assume that the solution of (3) or (4) can be expressed as a series in p, due to the fact that $0 \le p \le 1$ can be considered as small as parameter [21]. This means that we can write the solution of (3) or (4) as

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots . \tag{5}$$

For $p \to 1$, (3) or (4) corresponds to (2), hence (5) becomes the approximate solution of (2), i. e.,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots . \tag{6}$$

Series in (6) converges for most cases and so the rate of convergence depends on $A(v)$ [12].

3. Applications

In this section we apply HPM to solve some stiff systems of ordinary differential equations.

3.1. Problem 1

Consider the nonlinear initial value problem [22]

$$\begin{cases} y_1' = -1002\, y_1 + 1000\, y_2^2, \\ y_2' = y_1 - y_2(1 + y_2), \end{cases} \tag{7}$$

with the initial conditions

$$\begin{aligned} y_1(0) = 1, \\ y_2(0) = 1. \end{aligned} \tag{8}$$

Darvishi et al. [23] solved this system by the variational iteration method. According to HPM, we can construct a homotopy of system (7) as

$$\begin{aligned}(1-p)\big[v_1' - y_1'(0)\big] + p\big[v_1' + 1002 v_1 - 1000 v_1^2\big] = 0, \\ (1-p)\big[v_2' - y_2'(0)\big] + p\big[v_2' - v_1 + v_2(1+v_2)\big] = 0,\end{aligned} \tag{9}$$

where "prime" denotes differentiation with respect to x and initial approximations are as follows:

$$\begin{aligned} v_{1,0} = y_1(0), \\ v_{2,0} = y_2(0), \end{aligned} \tag{10}$$

and

$$\begin{aligned} v_1 = v_{1,0} + pv_{1,1} + p^2 v_{1,2} + p^3 v_{1,3} + p^4 v_{1,4} + \cdots , \\ v_2 = v_{2,0} + pv_{2,1} + p^2 v_{2,2} + p^3 v_{2,3} + p^4 v_{2,4} + \cdots , \end{aligned} \tag{11}$$

where $v_{i,j} = v_{i,j}(t)$ $(i,j = 1,2,3,\cdots)$ are functions to be determined. Substituting (10) and (11) into (9) and arranging the coefficient of "p" powers, yields

$$\begin{aligned}
& p\big[1002 v_{1,0} - 1000 v_{2,0}^2 + v_{1,0}' + v_{1,1}'\big] + \\
& p^2\big[1002 v_{1,1} - 2000 v_{2,0} v_{2,1} + v_{1,2}'\big] + \\
& p^3\big[1002 v_{1,2} - 1000 v_{2,1}^2 - 2000 v_{2,0} v_{2,2} + v_{1,3}'\big] \\
& + \cdots = 0, \\
& p\big[-v_{1,0} + v_{2,0} + v_{2,0}^2 + v_{2,0}' + v_{2,1}'\big] + \\
& p^2\big[-v_{1,1} + v_{2,1} + 2 v_{2,0} v_{2,1} + v_{2,2}'\big] + \\
& p^3\big[-v_{1,2} + v_{2,1}^2 + v_{2,2} + 2 v_{2,0} v_{2,2} + v_{2,3}'\big] \\
& + \cdots = 0.
\end{aligned} \tag{12}$$

Therefore we have:

$$\begin{aligned}
& 1002 v_{1,0} - 1000 v_{2,0}^2 + v_{1,0}' + v_{1,1}' = 0, \\
& 1002 v_{1,1} - 2000 v_{2,0} v_{2,1} + v_{1,2}' = 0, \\
& 1002 v_{1,2} - 1000 v_{2,1}^2 - 2000 v_{2,0} v_{2,2} + v_{1,3}' = 0, \\
& \vdots
\end{aligned}$$

$$-v_{1,0} + v_{2,0} + v_{2,0}^2 + v_{2,0}' + v_{2,1}' = 0,$$
$$-v_{1,1} + v_{2,1} + 2v_{2,0}v_{2,1} + v_{2,2}' = 0,$$
$$-v_{1,2} + v_{2,1}^2 + v_{2,2} + 2v_{2,0}v_{2,2} + v_{2,3}' = 0,$$
$$\vdots \tag{13}$$

From (6), if approximations by k terms be sufficient, we obtain

$$y_1(t) = \lim_{p\to 1} v_1(t) = \sum_{n=0}^{k} v_{1,n}(t), \tag{14}$$

$$y_2(t) = \lim_{p\to 1} v_2(t) = \sum_{n=0}^{k} v_{2,n}(t). \tag{15}$$

To calculate the terms of the homotopy series (14) and (15) for $y_1(t)$ and $y_2(t)$, we substitute the initial conditions (8) and (5) into the system (13) and finally we use Mathematica. Thus the solution of the equations can be obtained as follows:

$$v_{1,0} = y_1(0) = 1, \tag{16}$$

$$v_{1,1} = -2t, \tag{17}$$

$$v_{1,2} = 2t^2, \tag{18}$$

$$v_{1,3} = -\frac{4t^3}{3}, \tag{19}$$

$$\vdots$$

$$v_{2,0} = y_2(0) = 1, \tag{20}$$

$$v_{2,1} = -t, \tag{21}$$

$$v_{2,2} = \frac{t^2}{2}, \tag{22}$$

$$v_{2,3} = -\frac{t^3}{6}, \tag{23}$$

$$\vdots$$

The other components can be easily obtained in a similar manner. Substituting (17)–(24) into (14) and (15) yields

$$y_1(t) = 1 - 2t + 2t^2 - \frac{4t^3}{3} + \cdots + \frac{(-2t)^k}{k!} \tag{24}$$

and

$$y_2(t) = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots + \frac{(-t)^k}{k!}. \tag{25}$$

Using Taylor series, we obtain the closed form solutions as

$$y_1(t) = \exp(-2t),$$
$$y_2(t) = \exp(-t),$$

which are the exact solutions of the system.

3.2. Problem 2

Consider the nonlinear system of differential equations [24]

$$\begin{cases} y_1' = \lambda y_1 + y_2^2, \\ y_2' = -y_2, \end{cases} \tag{26}$$

where $\lambda = 10000$. The initial conditions of (26) are

$$y_1(0) = -1/(\lambda + 2),$$
$$y_2(0) = 1.$$

We can construct the following homotopy of system (26):

$$\begin{aligned} &(1-p)\big[v_1' - y_1'(0)\big] + p\big[v_1' - \lambda v_1 - v_2^2\big] = 0,\\ &(1-p)\big[v_2' - y_2'(0)\big] + p\big[v_2' + v^2\big] = 0. \end{aligned} \tag{27}$$

By substituting (10) and (11) into (27) and arranging the coefficient of "p" powers, we have

$$\begin{aligned} &p\big[-\lambda v_{1,0} - v_{2,0}^2 + v_{1,0}' + v_{1,1}'\big] +\\ &p^2\big[-\lambda v_{1,1} - 2v_{2,0}v_{2,1} + v_{1,2}'\big] +\\ &p^3\big[-\lambda v_{1,2} - v_{2,1}^2 - 2v_{2,0}v_{2,2} + v_{1,3}'\big] + \cdots = 0,\\ &p\big[v_{2,0} + v_{2,0}' + v_{2,1}'\big] + p^2\big[v_{2,1} + v_{2,2}'\big] +\\ &p^3\big[v_{2,2} + v_{2,3}'\big] + \cdots = 0. \end{aligned} \tag{28}$$

We set $v_{1,0} = y_1(0) = -1/(\lambda+2)$ and $v_{2,0} = y_2(0) = 1$. Then we obtain the following results:

$$v_{1,0} = -\frac{1}{2+\lambda}, \tag{29}$$

$$v_{1,1} = \frac{2t}{2+\lambda}, \tag{30}$$

$$v_{1,2} = -\frac{2t^2}{2+\lambda}, \tag{31}$$

$$v_{1,3} = \frac{4t^3}{3(2+\lambda)}, \tag{32}$$

$$\vdots$$

Table 1. Absolute errors of the approximations of y_1 for test problem 1 by k term approximations.

t	$k = 10$	$k = 20$	$k = 50$
0.1	5.5511, −16	0.0000	0.0000
0.2	1.0166, −12	2.2204, −16	2.2204, −16
0.5	2.3114, −08	1.1102, −16	1.1102, −16
1.0	4.3905, −05	3.7692, −14	5.5511, −17
1.5	3.5388, −03	1.8006, −10	4.1633, −17
2.0	7.8404, −02	7.2764, −08	2.6368, −16
2.5	8.5730, −01	7.5931, −06	2.4173, −15

Table 2. Absolute errors of the approximations of y_2 for test problem 1 by k term approximations.

t	$k = 10$	$k = 20$	$k = 50$
0.1	1.1102, −16	1.1102, −16	
0.2	5.5511, −16	0.0000	0.0000
0.5	1.1741, −11	1.1102, −16	1.1102, −16
1.0	2.3114, −08	1.1102, −16	1.1102, −16
1.5	1.9243, −06	5.5511, −17	5.5511, −17
2.0	4.3905, −05	3.7692, −14	5.5511, −17
2.5	4.9312, −04	3.9944, −12	2.7755, −17

$$v_{2,0} = 1, \tag{33}$$

$$v_{2,1} = -t, \tag{34}$$

$$v_{2,2} = \frac{t^2}{2}, \tag{35}$$

$$v_{2,3} = -\frac{t^3}{6}, \tag{36}$$

$$\vdots$$

The other components can be easily obtained in a similar manner. Substituting (30)–(37) into (14) and (15) yields

$$y_1(t) = -\frac{1}{2+\lambda}\left(1-2t+2t^2-\frac{4t^3}{3}+\cdots+\frac{(-2t)^k}{k!}\right) \tag{37}$$

and

$$y_2(t) = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots + \frac{(-t)^k}{k!}. \tag{38}$$

Using the Taylor series, we obtain the closed form solution as

$$y_1(t) = -\frac{\exp(-2t)}{2+\lambda},$$
$$y_2(t) = \exp(-t),$$

which are the exact solution of system (26).

3.3. Problem 3

We consider a system representing a nonlinear reaction that was taken by Hull et al. [25]

$$\begin{cases} y_1' = -y_1, \\ y_2' = y_1 - y_2^2, \\ y_3' = y_2^2. \end{cases} \tag{39}$$

The initial conditions are given by

$$y_1(0) = 1,$$
$$y_2(0) = 0,$$
$$y_3(0) = 0.$$

According to HPM we obtain the following results:

$$y_1(t) = 1 - t + \tfrac{t^2}{2} - \tfrac{t^3}{6} + \cdots, \tag{40}$$

$$y_2(t) = t - \tfrac{t^2}{2} - \tfrac{t^3}{3!} + \tfrac{5t^4}{4!} + \tfrac{4t^5}{5!} + \cdots, \tag{41}$$

$$y_3(t) = \tfrac{2t^3}{3!} - \tfrac{6t^4}{4!} - \tfrac{2t^5}{5!} + \cdots. \tag{42}$$

4. Numerical Results and Discussion

In this section, we obtain numerical solution for one of the previous examples. In order to verify the efficiency of the proposed method in comparison with the exact solution, we report the absolute errors for different values of t. For the computational work we select test problem 1.

The differences between the k term approximate solution of HPM and the exact solution are shown in Tables 1 and 2. We can see a very good agreement between the results of HPM and the exact solution which confirms the validity of HPM.

5. Conclusion

In this paper, the homotopy perturbation method was used for finding exact or approximate solutions of stiff systems of ordinary differential equations with initial conditions. For the examples studied in this paper, this method does not require small parameters, so the limitation of the traditional perturbation methods can be eliminated. Therefore, the calculations in the homotopy perturbation method are simple and straightforward.

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